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NUMERICAL INTEGRATION OF THE EQUATION OF MOTION FOR SURFACE WAVES IN A MEDIUM WITH ARBITRARY VARIATION OF MATERIAL CONSTANTS

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ABSTRACT

The calculation of surface-wave dispersion is difficult when the waves propagate in media whose physical properties change with depth, and only a few solutions are available for fairly simple cases.

These computations may now be performed with the aid of high-speed computers, even for media whose material constants change arbitrarily with depth. The dispersion of both Love waves and Rayleigh waves has been obtained for such cases by the numerical specification of surface displacement followed by numerical solution of the equations of motion.

For example, with respect to the problem of Love waves, besides the ordinary boundary condition that the stress vanishes at the free surface, an extra condition is stated which requires that the displacement amplitude be unity at the surface. The equation of motion is then solved numerically for tentative values of frequency and wave number, and this solution produces the distribution of displacement amplitude in the half space. For all combinations of frequency and wave number which are not solutions, the values of computed displacement do not converge and tend to become positively or negatively infinite for increasing depth below the free surface. To obtain a solution, one of the parameters—for instance, wave number—is fixed, and frequency is varied in small steps until the computed displacement converges to zero at great depths. This combination of parameters fulfills all the standard boundary conditions and is the required solution. The problem of sound waves in an elastic liquid can also be solved with only a minor change in the physical properties.

The dispersion of Rayleigh waves propagating in a heterogeneous substance can also be obtained by a similar method. In this case, another parameter is needed, namely, the ratio of the amplitude of horizontal and vertical components of displacement at the free surface. Denote this quantity by a, and the phase velocity by c. Wave number is fixed, and a two-dimensional search in the a–c plane is used to locate the point that produces a convergent solution. For limited media a solution is required which satisfies the boundary conditions at the other surface.

After testing the method by applying it to cases which had been solved analytically, a few problems were solved. These include:

1) Love waves in a medium with constant density and linearly increasing rigidity.

2) Sound waves in a medium whose density and velocity are given by experimental curves.

3) Rayleigh waves in a medium having constant density and equal rates of increase for λ and μ.

Using an IBM "650," it takes a few minutes to get a point for cases 1 and 2, and from thirty minutes to two hours per point for Rayleigh-wave dispersion.

Introduction

Determination of the velocity of propagation is one of the most interesting as well as important problems in the study of the propagation of surface waves. When the medium consists of one or two homogeneous materials with layered structure, it is not difficult to calculate the dispersion curve. Today, by the use of electronic computing machines, it has become possible for us to assume a large number of layers, every one of which is homogeneous.[1]

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[1] James Dorman, personal communication.

If the material is heterogeneous, however, the analytical method of solution becomes very difficult. The dispersion curve has been computed for certain structures having relatively simple variation of material properties,[2] and various methods of approximation have been suggested,[3] but not without restrictions such as the continuous increase of the body-wave velocity.

The numerical method presented here has no restrictions in the distribution of the material. The problem is assumed to be two-dimensional.

1. Simple Love waves

First, we shall consider a simple problem of Love waves, which can be easily solved analytically.

The horizontal displacement v perpendicular to the x–z plane satisfies the differential equation

$$\nabla^2 v = \frac{1}{c_S{}^2} \frac{\partial^2 v}{\partial t^2} \tag{1.1}$$

where c_S is the velocity of S waves.

Assuming the form of solution

$$v = V(z) \exp (ipt - ifx) \tag{1.2}$$

substitute into the foregoing equation. The equation for $V(z)$ becomes

$$\frac{d^2V}{dz^2} + (p^2/c_S{}^2 - f^2)V = 0 \tag{1.3}$$

The solution is given by

$$\begin{aligned} V(z) &= A \cos \tilde{\beta} z + B \sin \tilde{\beta} z \\ \tilde{\beta} &= \sqrt{p^2/c_S{}^2 - f^2} \end{aligned} \tag{1.4}$$

Similarly in the half space the differential equation for the displacement v' is

$$\nabla^2 v' = \frac{1}{c_{S'}{}^2} \frac{\partial^2 v'}{\partial t^2} \tag{1.5}$$

and the function $V'(z)$, which gives the vertical distribution of the amplitude, satisfies the equation

$$\frac{d^2V'}{dz^2} - \left(f^2 - \frac{p^2}{c_{S'}{}^2}\right) V' = 0 \tag{1.6}$$

where $c_{S'}$ implies the shear velocity in the half space.

[2] Cf. M. Ewing, W. Jardetzky, and F. Press, *Elastic Waves in Layered Media* (New York: McGraw-Hill, 1957), chap. vii.

[3] E.g., H. Jeffreys, *Proc. London Math. Soc.*, 23: 428–436 (1923); C. L. Pekeris, *Physics*, 6: 133–138 (1935); M. Newlands, *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, 6: 109–125 (1950); T. Takahashi, *Bull. Earthq. Res. Inst.*, Tokyo Univ., 33: 287–296 (1955), and 35: 297–308 (1957).

The solution of this equation is

$$V'(z) = A' \exp(\beta' z) + B' \exp(-\beta' z)$$
$$\beta' = \sqrt{f^2 - p^2/c_{S'}^2} \tag{1.7}$$

The condition at the free surface $z = -H$ is

$$p_{yz} = \mu \frac{\partial v}{\partial z} = 0 \tag{1.8}$$

at the surface of separation of two media $z = 0$,

$$p_{yz} = p_{yz'}$$
and $$v = v' \tag{1.9}$$

In addition to these conditions we usually assume that v' vanishes at $z = \infty$. In place of this last condition, here we will instead assume that the amplitude is C at the free surface, that is,

$$A \cos \tilde{\beta} H - B \sin \tilde{\beta} H = C \tag{1.10}$$

By means of these four equations (1.8), (1.9), and (1.10) we can determine the four quantities A, B and A', B'.

Omitting A and B, which are not necessary now, we will give the expressions for A' and B'. A' must be zero in order to get a convergent solution at $z = \infty$.

$$\begin{cases} A' = C\left(\cos \tilde{\beta} H - \dfrac{\mu \tilde{\beta}}{\mu' \beta'} \sin \tilde{\beta} H\right)/2 \\ B' = C\left(\cos \tilde{\beta} H + \dfrac{\mu \tilde{\beta}}{\mu' \beta'} \sin \tilde{\beta} H\right)/2 \end{cases} \tag{1.11}$$

If we put A' equal to zero, we have the ordinary characteristic equation for Love waves; or, in other words, this equation is the condition that the displacement vanishes at $z = \infty$. If, however, this relation does not hold, the term $A' \exp(\beta' z)$ goes to $+\infty$ or $-\infty$ according to the sign of A. Only the proper combination of p and f makes A' zero and the solution convergent.

Suppose we solve the differential equations (1.3) and (1.6) numerically, starting from the free surface under the conditions (1.8), (1.9), and (1.10). In this case we must tentatively give the values of p and f in order to start a numerical solution. Generally they do not provide a correct answer and the solution will not be convergent. However, if we fix f and change p by small steps (or fix p and change f), we shall finally get to the right value of p (or f), which gives the convergent solution of the differential equation in the half space. The ratio p/f gives the phase velocity, and the integrated curve of the equation gives the distribution of the amplitude. Figure 1 shows an example of the calculation and illustrates how the distribution of the amplitude changes with the various combinations of p and f.

2. Love waves in a heterogeneous medium

Even if the medium contains no layers, surface waves of the Love-wave type can exist under certain conditions.[4]

The fundamental equation for this case is, since the x and z components of displacement are missing,

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) \tag{2.1}$$

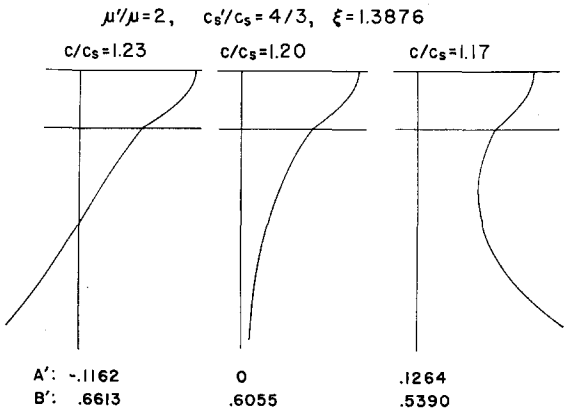

Fig. 1. When $\mu'/\mu = 2$, $c_S'/c_S = 4/3$, corresponding to $\xi = 1.3876$, we have $c/c_S = 1.20$ as the root of the characteristic equation for Love waves. That means A' in (1.11) becomes zero for this value of c/c_S and the solution of the differential equation (1.6) converges to zero (middle figure). However, if $c/c_S = 1.23$, A' becomes negative and the solution of the differential equation diverges to $-\infty$ (left figure), while if $c/c_S = 1.17$, A' is positive and the curve goes to $+\infty$ (right figure).

If μ is a function of z only, (2.1) becomes

$$\rho \frac{\partial^2 v}{\partial t^2} = \mu \nabla^2 v + \frac{d\mu}{dz}\frac{\partial v}{\partial z} \tag{2.2}$$

If we assume that

$$v = V_0(z) \exp(ipt - ifx) \tag{2.3}$$

$V_0(z)$ will be the solution of the equation

$$V_0'' + \frac{\mu'}{\mu} V_0' + \left(p^2 \frac{\rho}{\mu} - f^2\right) V_0 = 0 \tag{2.4}$$

[4] Existence has often been proved practically, by giving the solution. The theoretical discussion of the problem of existence is found in Z. Suzuki, "On Love Waves in Heterogeneous Media," *Science Reports*, Tôhoku Univ., ser. 5, 7: 82–93 (1955).

This equation can be reduced to a simpler form

$$V'' + \left[M(z) + \left(p^2 \frac{\rho}{\mu} - f^2\right)\right] V = 0$$
$$M(z) = \frac{\mu'^2}{4\mu^2} - \frac{\mu''}{2\mu} \tag{2.5}$$

using a substitution $V_0(z) = \mu^{-1/2} V(z)$. We will simply denote this equation by

$$L(p, f; z) = 0 \tag{2.6}$$

The boundary condition at the free surface is

$$-\frac{\mu'}{2\mu} V + V' = 0 \tag{2.7}$$

Suppose the distribution of the material gives a favorable condition under which some kind of surface wave can exist. If the phase velocity of this wave corresponding to the frequency p is c, the differential equation

$$L(p, p/c; z) = 0 \tag{2.8}$$

must have a convergent solution satisfying the boundary condition (2.7). If the phase velocity differs from c by a small amount Δc, the solution of

$$L(p, p/\overline{c + \Delta c}\ ; z) = 0 \tag{2.9}$$

satisfying (2.7) does not generally give a convergent solution, because the phase velocity cannot take an arbitrary value if the frequency is determined. If, therefore, there is a solution with a frequency p, we can find it by substituting various values of c into the differential equation (2.8) until we find a convergent solution which satisfies (2.7). The simple Love-wave computation described in the preceding section is an example of the use of this process. For that case the solution can be easily obtained analytically, but even if the analytical solution is difficult or impossible because of a complicated distribution of a material, we can still solve the equation numerically by using the method described above.

2.1. Linearly increasing density and rigidity

Meissner[5] first considered the problem of a half space in which the density and the rigidity are given by the expressions

$$\rho = \rho_0(1 + \delta z), \qquad \mu = \mu_0(1 + \epsilon z) \tag{2.9}$$

This problem can be easily reduced to that of $\delta = 0$, or constant density.

Introducing (2.9) into (2.5) we have

$$\frac{d^2V}{d\zeta^2} + \left[\frac{1}{4\zeta^2} + \frac{\omega^2}{\zeta}\left(1 - \frac{\delta}{\epsilon}\right) - \left(\xi^2 - \omega^2 \frac{\delta}{\epsilon}\right)\right] V = 0 \tag{2.10}$$

[5] E. Meissner, "Elastische Oberflächenwellen mit Dispersion in einem inhomogenen Medium," *Vierteljahresschr. Naturforsch. Ges.*, Zürich, 66: 181–195 (1921), and "Elastische Oberflächen-Querwellen," *Proc. Sec. Int. Cong. Appl. Mech.* (Zürich), 3–11 (1926).

where

$$\zeta = 1 + \epsilon z$$

$$\xi = f/\epsilon\,, \quad \omega^2 = p^2 \frac{\rho_0}{\mu_0} \frac{1}{\epsilon^2}$$

If we put

$$\omega^2(1 - \delta/\epsilon) \to \omega^2\,, \qquad \xi^2 - \omega^2\delta/\epsilon \to \xi^2 \tag{2.11}$$

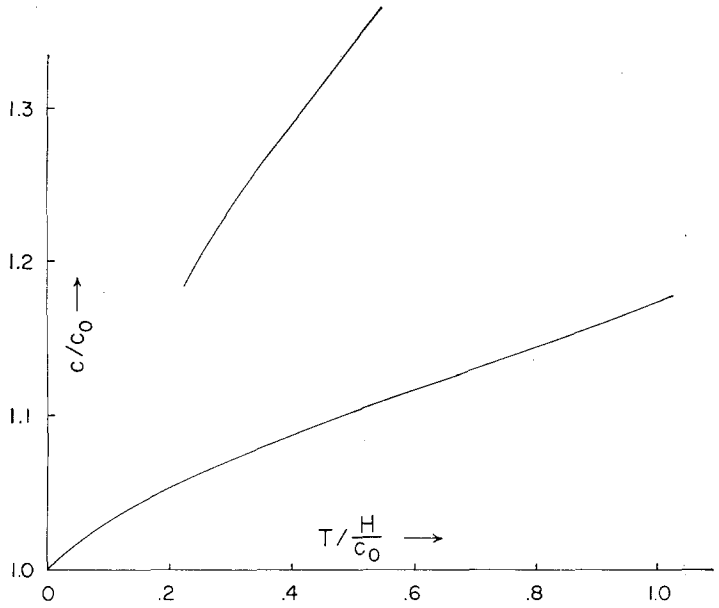

Fig. 2. Phase velocity of first- and second-mode Love waves in a half space with linearly increasing shear modulus and constant density.

into the foregoing expression we have

$$\frac{d^2V}{d\zeta^2} + \left[\frac{1}{4\zeta^2} + \frac{\omega^2}{\zeta} - \xi^2\right] V = 0 \tag{2.12}$$

which is identical to the equation for a medium with constant density.

This equation (2.12) can be solved following the principle given in section 2 under the condition of zero stress at the free surface of the half space. The numerical result is given in figure 2.

The following simple formula was employed for the numerical solution of the equation

$$V_{n+1} = V_n + \Delta V_{n-1} + h^2 V_n'' + O(h^4) \tag{2.13}$$

where h is the interval of two consecutive variables, and

$$\Delta V_{n-1} = V_n - V_{n-1}$$

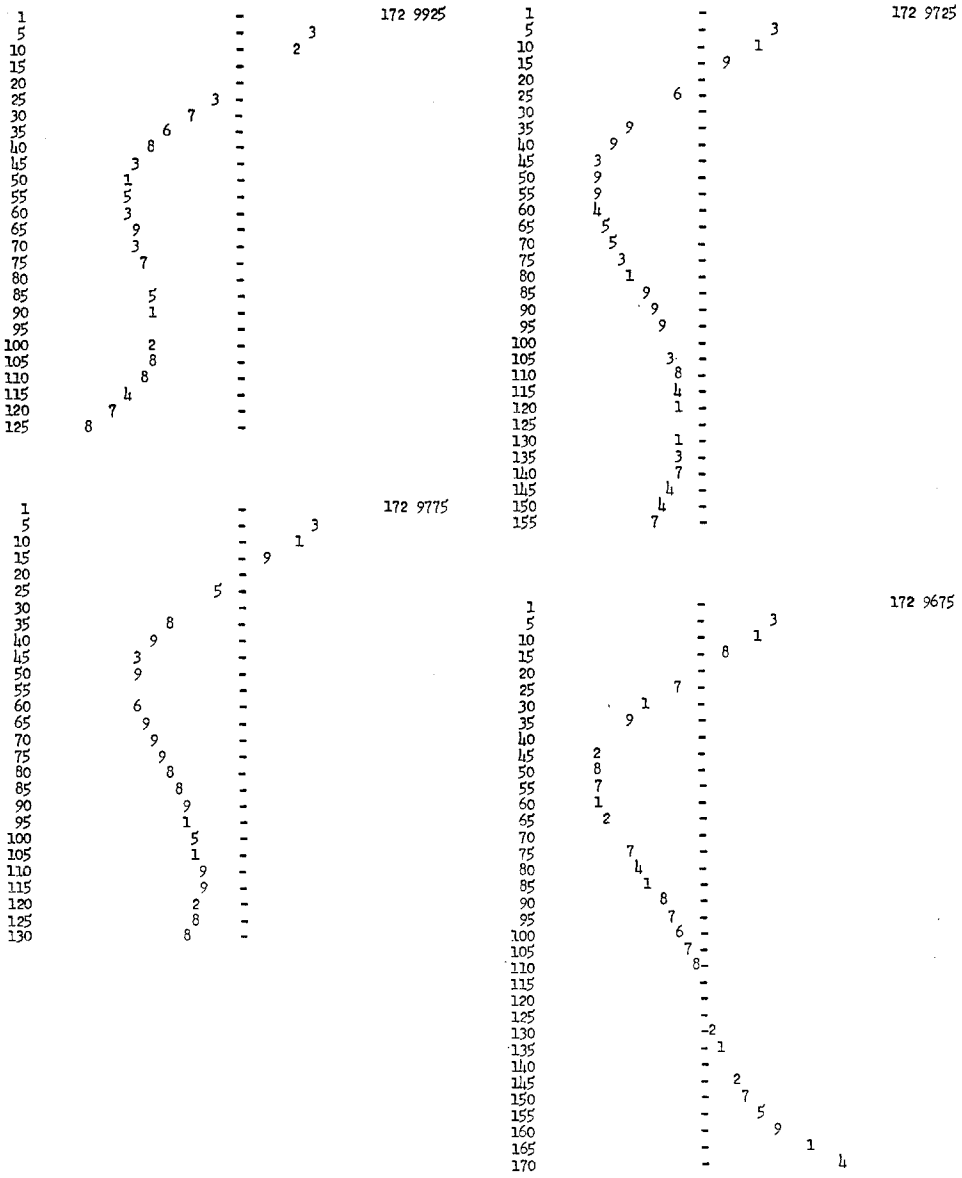

Fig. 3. In order to save time required for plotting, an IBM "CPC" was used to draw a curve given as the solution of the differential equation (2.12). The minus signs denote the zero line. The value of V is given to six digits, say, 1.23456. Only the first three digits 1.23 are adopted for plotting. The third number, "3," is printed at the point indicated by the first two numbers, namely, the twelfth location to the right of the zero line. Zeroes are not printed by the machine. Two parameters are shown on the right side of the figure. The first parameter (172) was fixed and the second was changed. The correct answer lies between 9725 and 9675 in this case.

An IBM "607" electronic computer was used for this computation. Using the substitution of (2.11) the solution in a medium with linearly increasing density can be obtained. Parts of the first-mode curve have been published by a number of authors.[6] Figure 3 shows how we can determine the range of one parameter while the other is fixed when computing the second mode.

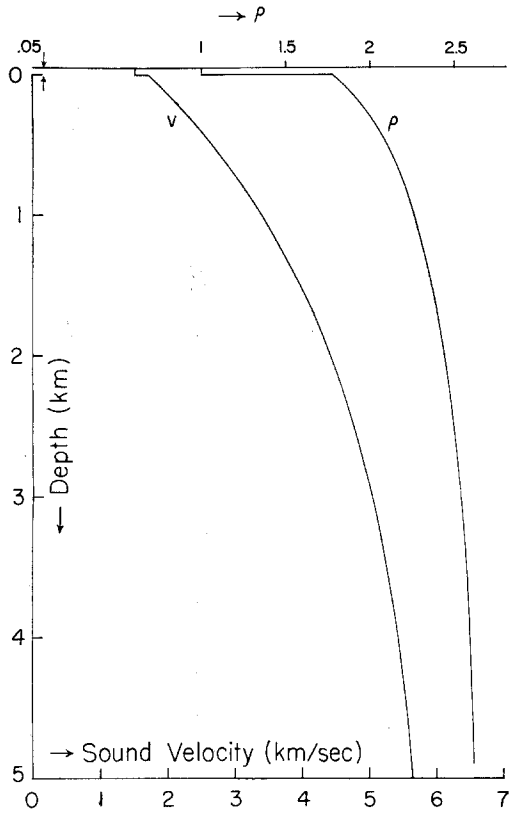

Fig. 4. Density and sound velocity distribution with depth for C. Drake's case.

3. Sound waves in a heterogeneous liquid medium

When the medium is an elastic liquid we have a formula similar to the previous one. The fundamental equation in a two-dimensional liquid medium is

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}(\lambda\Delta) \\ \rho \dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial z}(\lambda\Delta), \qquad \Delta = \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} \end{cases} \tag{3.1}$$

If we put

$$-\lambda\Delta = \tilde{w} \tag{3.2}$$

[6] First paper of T. Takahashi cited in note 3 above.

which is the pressure, into the foregoing equation we have

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = -\dfrac{\partial \tilde{w}}{\partial x} \\ \rho \dfrac{\partial^2 w}{\partial t^2} = -\dfrac{\partial \tilde{w}}{\partial z} \end{cases} \tag{3.3}$$

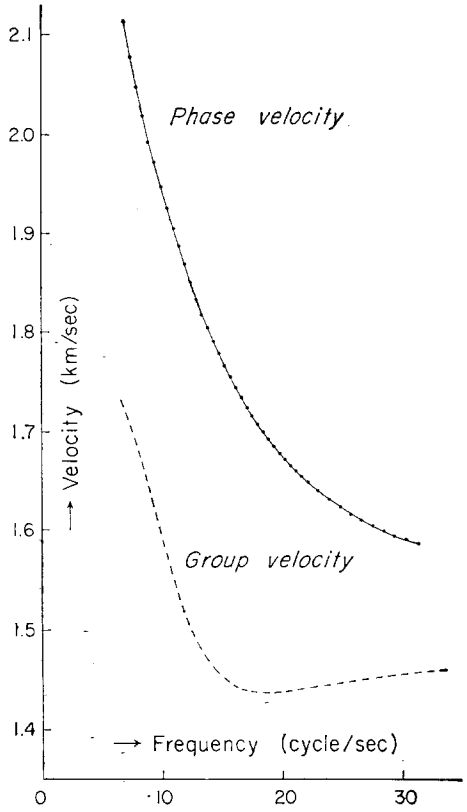

Fig. 5. Calculated phase and group velocity based on the data given in figure 4. Group velocity was obtained by numerically differentiating the phase velocity.

From these equations we can easily get to the next one

$$\rho \frac{\partial^2 \tilde{w}}{\partial t^2} = \lambda \nabla^2 \tilde{w} - \lambda \frac{\rho'}{\rho} \frac{\partial \tilde{w}}{\partial z} \tag{3.4}$$

If we make the substitutions

$$\begin{aligned} \rho &\text{—} 1/\mu \\ \lambda &\text{—} 1/\rho \end{aligned} \tag{3.5}$$

and

$$\tilde{w} \text{—} v$$

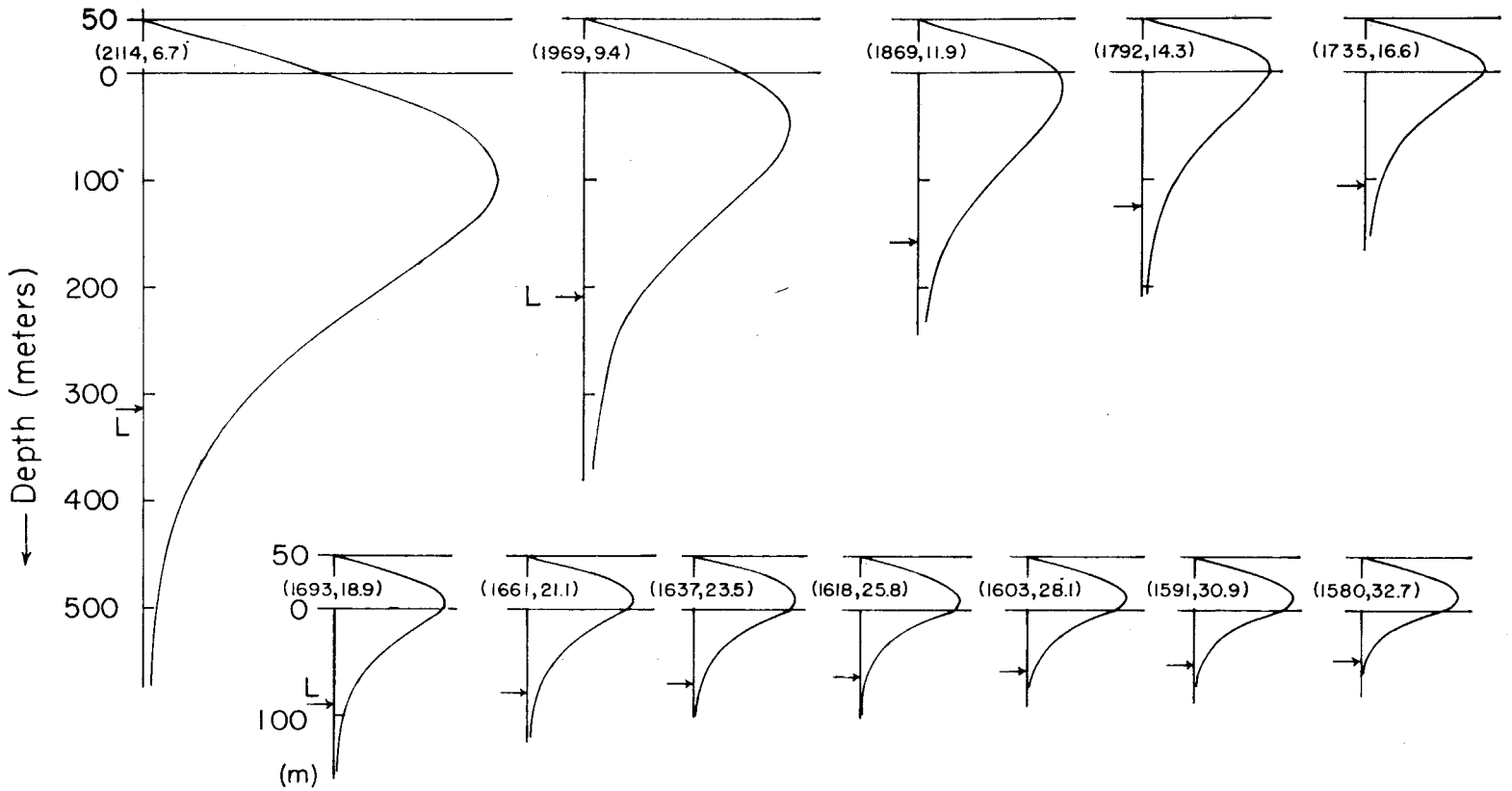

Fig. 6. In the half space, the curve shows the distribution of Π, which is roughly proportional to the pressure because the variation of density is not large. In the surface layer $[\rho \text{ (layer)}/\rho \text{ (surface value in the half space)}]^{\frac{1}{2}} \Pi$ is shown, which is proportional to the pressure. The numbers in parentheses are, right, frequency in cycles/sec., left, phase velocity in m/sec.; the letter L and the arrow indicate the depth equal to one wave length for each curve.

into this expression we have an equation identical to (2.2), which is the equation for the SH wave problem.

Also for the boundary conditions there is a correspondence between the two problems.

At the free surface of the liquid medium we have $\tilde{w} = 0$, which corresponds, from (3.5), to $v = 0$ in the SH wave problem. The rigid boundary of a liquid medium requires $w = 0$, which is identical to $\partial\tilde{w}/\partial z = 0$ from (3.3) and corresponds to the free surface in the case of SH waves. Summarizing these cases we have the relations:

$$\begin{array}{ll} \text{Sound waves} & \text{SH waves} \\ \text{Free surface} & \text{Fixed surface} \\ \text{Rigid surface} & \text{Free surface} \end{array} \tag{3.6}$$

If the substitutions given by (3.5) and (3.6) are made, a problem of sound waves turns out to be that of SH waves, and vice versa. Therefore, the fundamental equation of this case can be obtained by slightly modifying the equation (2.5). Putting

$$\begin{aligned} \tilde{w} &= \Pi_0(z) \exp(ipt - ifx) \\ \Pi_0 &= \rho^{1/2}\Pi \end{aligned} \tag{3.7}$$

we have

$$\begin{aligned} \Pi'' + \left[R(z) + \left(p^2\frac{\rho}{\lambda} - f^2\right)\right]\Pi &= 0 \\ R(z) &= -\frac{3}{4}\frac{\rho'^2}{\rho^2} + \frac{1}{2}\frac{\rho''}{\rho} \end{aligned} \tag{3.8}$$

3.1. Example of sound waves in a heterogeneous liquid

An interesting example based on the data from shallow-water explosions has been presented by Charles Drake of the Lamont Geological Observatory.[7] The distribution of sound velocity and density is shown in figure 4.

Although there is a surface layer of homogeneous water, there is no essential change of the principle of calculation. The result, which was obtained with the aid of an IBM "650," is given in figure 5. A few minutes are required to compute a point on the dispersion curve. The group velocity was obtained by numerical differentiation. In figure 6 the distribution of Π is given, which shows how the pressure changes with depth.

4. Rayleigh waves

In the previous sections we have dealt with the problems of SH waves in a solid and sound waves in a liquid. We shall now develop our method of solving the problem of Rayleigh waves, for which there are two variables, dilatation and rotation or horizontal and vertical displacements. The number of parameters also increases in this case.

[7] C. Drake, personal communication.

4.1. Theory of simple Rayleigh waves

First, assume a homogeneous medium in which the motion can be expressed by two potentials ϕ and ψ satisfying differential equations of the type

$$\nabla^2\phi = \frac{1}{c_P{}^2}\frac{\partial^2\phi}{\partial t^2}, \qquad \nabla^2\psi = \frac{1}{c_S{}^2}\frac{\partial^2\psi}{\partial t^2} \tag{4.1}$$

respectively. c_P and c_S are the velocities of P and S waves. When the direction of z positive is taken vertically downwards, the two components of motion, u and w, may be expressed in terms of ϕ and ψ as

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \qquad w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{4.2}$$

Since at first we assume the medium to be homogeneous, the equation (4.1) is satisfied by the functions

$$\begin{cases} \phi = \{A\exp(\alpha z) + B\exp(-\alpha z)\}\exp(ipt - ifx) \\ \psi = \{C\exp(\beta z) + D\exp(-\beta z)\}\exp(ipt - ifx) \\ \alpha^2 = f^2 - p^2/c_P{}^2, \qquad \beta^2 = f^2 - p^2/c_S{}^2 \end{cases} \tag{4.3}$$

The stress components p_{xz} and p_{zz}, which must vanish at the free surface $z = 0$, then take the following form

$$\begin{cases} p_{xz} = \mu\left(2\dfrac{\partial^2\phi}{\partial x\partial z} + \dfrac{\partial^2\psi}{\partial x^2} - \dfrac{\partial^2\psi}{\partial z^2}\right) \\ \quad = \mu[A(-i2f\alpha) + B(i2f\alpha) + C(-2f^2 + p^2/c_S{}^2) + D(-2f^2 + p^2/c_S{}^2)] \\ p_{zz} = \lambda\nabla^2\phi + 2\mu\left(\dfrac{\partial^2\phi}{\partial z^2} + \dfrac{\partial^2\psi}{\partial x\partial z}\right) \\ \quad = \mu[A(2f^2 - p^2/c_S{}^2) + B(2f^2 - p^2/c_S{}^2) + C(-i2f\beta) + D(i2f\beta)] \end{cases} \tag{4.4}$$

In addition to the two boundary conditions

$$p_{xz} = 0, \qquad p_{zz} = 0 \tag{4.5}$$

at $z = 0$, we usually assume two extra conditions

$$u = w = 0, \qquad \text{at} \quad z = \infty$$

In this case, however, we will assume the following conditions in place of the foregoing two:

$$\begin{cases} iu = aQ \exp(ipt - ifx) \\ w = Q \exp(ipt - ifx) \end{cases} \tag{4.6}$$

where a is the ratio of the amplitude of two displacement components.

We have four parameters A, B and C, D in (4.3), and the four conditions in (4.5) and (4.6) determine the values of all four quantities, namely

$$\begin{cases} A = K\{-(2f^2 - p^2/c_S^2) + 2af\alpha\}/\alpha \\ B = K\{(2f^2 - p^2/c_S^2) + 2af\alpha\}/\alpha \\ iC = K\{a(2f^2 - p^2/c_S^2) - 2f\beta\}/\beta \\ iD = K\{-a(2f^2 - p^2/c_S^2) - 2f\beta\}/\beta \end{cases} \tag{4.7}$$

$$K = Q(c_S^2/2p^2)$$

Since A is the coefficient of the exponential function with a positive argument, A must be zero so long as we search for a solution convergent at $z = \infty$. The same situation holds with respect to C and we must have

$$\begin{cases} A = 0 \\ C = 0 \end{cases} \quad \text{or} \quad \begin{cases} -(2f^2 - p^2/c_S^2) + 2af\alpha = 0 \\ a(2f^2 - p^2/c_S^2) - 2f\beta = 0 \end{cases} \tag{4.8}$$

These are the two equations involving two unknowns a and c. They give the surface-wave velocity and also the ratio of the amplitude of the two components of motion which vanish at $z = \infty$. If we eliminate a from (4.8), we have the ordinary characteristic equation for Rayleigh waves

$$(2f^2 - p^2/c_S^2)^2 - 4f^2\alpha\beta = 0 \tag{4.9}$$

The first equation of (4.8) defines a curve X in the a–c plane and the second equation defines curve Y. The intersection of the two curves, point P, gives the velocity and the amplitude ratio of Rayleigh waves.

5. Numerical method for Rayleigh waves

Suppose there is a set of differential equations for u amd w

$$L_1(u, w) = 0, \qquad L_2(u, w) = 0 \tag{5.1}$$

After specifying the conditions at the free surface we can solve the equations numerically starting from $z = 0$. At the beginning, however, we must give the values of a and c tentatively. In the case of the homogeneous half space, if the values of a

and c give the coördinates of P in the a–c plane in figure 7, then the solution converges to zero at $z = \infty$ and gives the true distribution of u and w. At other points in the a–c plane, A and/or C take values other than zero and the solution does not converge at $z = \infty$.

This method of searching for the combination of two parameters can be applied to the problem of heterogeneous Rayleigh waves. If the solution exists, it requires a certain definite combination of parameters, which must be obtained by trying the solution of the equation with various combinations of parameters until a correct result is found.

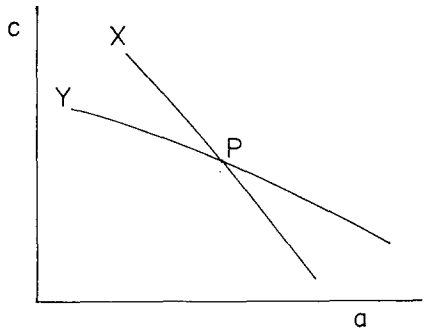

Fig. 7. Curve X satisfies the condition that A, the first expression in (4.7) vanishes, while the curve Y satisfies the condition that C becomes zero. The intersection P gives the velocity and the amplitude ratio which provide the solution of equation (4.1), which is the Rayleigh-wave solution.

The equation of motion when the medium is not necessarily homogeneous is as follows:

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left\{(\lambda + 2\mu)\dfrac{\partial u}{\partial x} + \lambda \dfrac{\partial w}{\partial z}\right\} + \dfrac{\partial}{\partial z}\left\{\mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right)\right\} \\ \rho \dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left\{\mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right)\right\} + \dfrac{\partial}{\partial z}\left\{(\lambda + 2\mu)\dfrac{\partial w}{\partial z} + \lambda \dfrac{\partial u}{\partial x}\right\} \end{cases} \tag{5.2}$$

Assuming the form

$$\begin{cases} u = U_0(z) \exp(ipt - ifx) \\ w = W_0(z) \exp(ipt - ifx) \end{cases} \tag{5.3}$$

we have differential equations for $U_0(z)$ and $W_0(z)$

$$\begin{cases} \mu \dfrac{d^2 U_0}{dz^2} + \dfrac{d\mu}{dz}\dfrac{dU_0}{dz} + f^2\left\{\dfrac{\rho p^2}{f^2} - (\lambda + 2\mu)\right\} U_0 - if\left\{(\lambda + \mu)\dfrac{dW_0}{dz} + \dfrac{d\mu}{dz} W_0\right\} = 0 \\ (\lambda + 2\mu)\dfrac{d^2 W_0}{dz^2} + \dfrac{d}{dz}(\lambda + 2\mu) \cdot \dfrac{dW_0}{dz} + f^2\left\{\dfrac{\rho p^2}{f^2} - \mu\right\} W_0 \\ \qquad - if\left\{(\lambda + \mu)\dfrac{dU_0}{dz} + \dfrac{d\lambda}{dz} U_0\right\} = 0 \end{cases} \tag{5.4}$$

Boundary conditions at the free surface are

$$\begin{cases} -if\,W_0 + \dfrac{dU_0}{dz} = 0 \\ -if\,U_0 + (1 + 2\mu/\lambda)\dfrac{dW_0}{dz} = 0 \end{cases} \tag{5.5}$$

Introducing a new notation

$$\gamma^2 \equiv (\lambda + 2\mu)/\mu \tag{5.6}$$

which may be a function of z, and also employing new independent variables U and W defined by

$$iU_0 = \mu^{-1/2}U \qquad \text{and} \qquad W_0 = \gamma_0(\lambda + 2\mu)^{-1/2}W \tag{5.7}$$

where

$$\gamma_0 = \gamma_{z=0}$$

we have the following equations of motion:

$$\begin{cases} U'' + \{M(z) + K(z)\}\,U + f(\gamma^2-1)\dfrac{\gamma_0}{\gamma}W' + f\dfrac{1}{\gamma^2}\left\{-\dfrac{\gamma^2-1}{2}\dfrac{\lambda'}{\mu} + \dfrac{\mu'}{\mu}\right\}\dfrac{\gamma_0}{\gamma}W = 0 \\ \dfrac{\gamma_0}{\gamma}W'' + \{N(z)+L(z)\}\dfrac{\gamma_0}{\gamma}W - f\dfrac{\gamma^2-1}{\gamma^2}U' + f\dfrac{1}{\gamma^2}\left\{-\dfrac{\lambda'}{\mu} + \dfrac{\gamma^2-1}{2}\dfrac{\mu'}{\mu}\right\}U = 0 \end{cases}$$

where

$$\begin{aligned} M(z) &= \frac{\mu'^2}{4\mu^2} - \frac{\mu''}{2\mu} & K(z) &= f^2\left(\frac{c^2}{c_S{}^2} - \gamma^2\right) \\ N(z) &= \frac{(\lambda' + 2\mu')^2}{4(\lambda + 2\mu)^2} - \frac{\lambda'' + 2\mu''}{2(\lambda + 2\mu)}, & L(z) &= f^2\left(\frac{c^2}{c_P{}^2} - \frac{1}{\gamma^2}\right) \end{aligned} \tag{5.8}$$

Boundary conditions that the surface is free from tractions are, from (5.5) and (5.7),

$$\begin{cases} f\dfrac{\gamma_0}{\gamma}W + U' - \dfrac{1}{2}\dfrac{\mu'}{\mu}U = 0 \\ -f\,U + \dfrac{\gamma_0\gamma}{\gamma^2 - 2}W' - \dfrac{1}{2}\dfrac{\gamma_0}{\gamma(\gamma^2-2)}\dfrac{\lambda' + 2\mu'}{\mu}W = 0 \end{cases} \tag{5.9}$$

In addition to the foregoing two conditions, which require the two stress components to be zero, we state two extra conditions corresponding to (4.6), namely,

$$iU_0 = aQ\,, \qquad W_0 = Q \tag{5.10}$$

at $z = 0$.

Since Q is arbitrary, we may change (5.10) to

$$U = a \qquad W = 1 \tag{5.11}$$

Using (5.8), (5.9), and (5.11), we can determine the distribution of U and W as functions of z with a parameter f, provided a and c are given tentatively.

If γ is constant, which requires that the ratio of the velocity of P and S waves remains constant, we obtain somewhat simpler expressions.

Equations of motion:

$$\begin{cases} U'' + \{M(z) + K(z)\}U + f(\gamma^2 - 1)W' - \frac{1}{2} f(\gamma^2 - 3)\frac{\mu'}{\mu} W = 0 \\ W'' + \{M(z) + L(z)\}\, W - f\frac{\gamma^2 - 1}{\gamma^2} U' - \frac{1}{2} f \frac{\gamma^2 - 3}{\gamma^2} \frac{\mu'}{\mu} U = 0 \end{cases}$$

$$M(z) = \frac{\mu'^2}{4\mu^2} - \frac{\mu''}{2\mu}, \qquad K(z) = f^2\left(\frac{c^2}{c_S{}^2} - \gamma^2\right) \tag{5.12}$$

$$L(z) = f^2\left(\frac{c^2}{c_P{}^2} - \frac{1}{\gamma^2}\right)$$

Boundary conditions:

$$\begin{cases} fW + U' - \frac{1}{2}\frac{\mu'}{\mu} U = 0 \\ -fU + \frac{\gamma^2}{\gamma^2 - 2} W' - \frac{\gamma^2}{\gamma^2 - 2}\frac{\mu'}{2\mu} W = 0 \end{cases} \tag{5.13}$$

Further, if λ, μ, and ρ are all constant, $M(z)$ vanishes and (5.12) reduces to a simpler form

$$\begin{cases} U'' + K \cdot U + f(\gamma^2 - 1)W' = 0 \\ W'' + L \cdot W - f\frac{\gamma^2 - 1}{\gamma^2} U' = 0 \end{cases} \tag{5.14}$$

where K and L are no longer functions of z.

(5.13) is also modified and becomes

$$\begin{cases} fW + U' = 0 \\ -fU + \frac{\gamma^2}{\gamma^2 - 2} W' = 0 \end{cases} \tag{5.15}$$

If we solve (5.14) under the conditions (5.11), (5.15), assuming that U and W both vanish at great depth, we should obtain the well-known values of the ratio of the two displacement components and the velocity of Rayleigh waves.

This method was tested and the correct values of a and c were obtained for various values of f, namely,

$$a = 0.6813 \qquad \text{and} \qquad c^2/c_S^2 = 0.845 \tag{5.16}$$

for the case $\lambda = \mu = \text{constant}$.

5.1. Rayleigh waves in a heterogeneous medium—linearly increasing elastic constants

Eqs. (5.8) and (5.9) are the general equations which can be used even when the material constants are arbitrary functions of depth. However, in order to compare with the case of Love waves which was discussed in section 2, we assume linearly increasing elasticity and constant density. γ was assumed to be constant.

In this case things are much simplified. Equation of motion:

$$\begin{cases} \dfrac{d^2U}{d\zeta^2} + \{\overline{M}(\zeta) + \overline{K}(\zeta)\}\, U + (\gamma^2 - 1)\,\xi \dfrac{dW}{d\zeta} - \dfrac{1}{2}(\gamma^2 - 3)\dfrac{\xi}{\zeta} W = 0 \\ \dfrac{d^2W}{d\zeta^2} + \{\overline{M}(\zeta) + \overline{L}(\zeta)\}\, W - \dfrac{\gamma^2 - 1}{\gamma^2}\,\xi \dfrac{dU}{d\zeta} - \dfrac{1}{2}\dfrac{\gamma^2 - 3}{\gamma^2}\dfrac{\xi}{\zeta} U = 0 \end{cases} \tag{5.17}$$

where

$$\mu = \mu_0(1 + \epsilon z)\,, \qquad \lambda/\mu = \gamma^2 - 2 = \text{const.}$$

$$\zeta = \mu/\mu_0 = 1 + \epsilon z\,, \qquad \mu_0 = \mu_{z=0}$$

and

$$\overline{M}(\zeta) = 1/4\zeta^2$$

$$\overline{K}(\zeta) = \left(\frac{\mathfrak{v}^2}{\zeta} - \gamma^2\right)\xi^2\,, \qquad \overline{L}(\zeta) = \left(\frac{1}{\gamma^2}\frac{\mathfrak{v}^2}{\zeta} - \frac{1}{\gamma^2}\right)\xi^2$$

$$\xi = f/\epsilon\,, \qquad \mathfrak{v} = c/c_0 = c/c_{S,\,z=0}$$

Boundary conditions:

$$(z = 0, \text{ or } \zeta = 1)$$

$$\begin{cases} \xi W + \dfrac{dU}{d\zeta} - \dfrac{1}{2\zeta} U = 0 \\ -\xi U + \dfrac{\gamma^2}{\gamma^2 - 2}\dfrac{dW}{d\zeta} - \dfrac{\gamma^2}{\gamma^2 - 2}\dfrac{1}{2\zeta} W = 0 \end{cases} \tag{5.18}$$

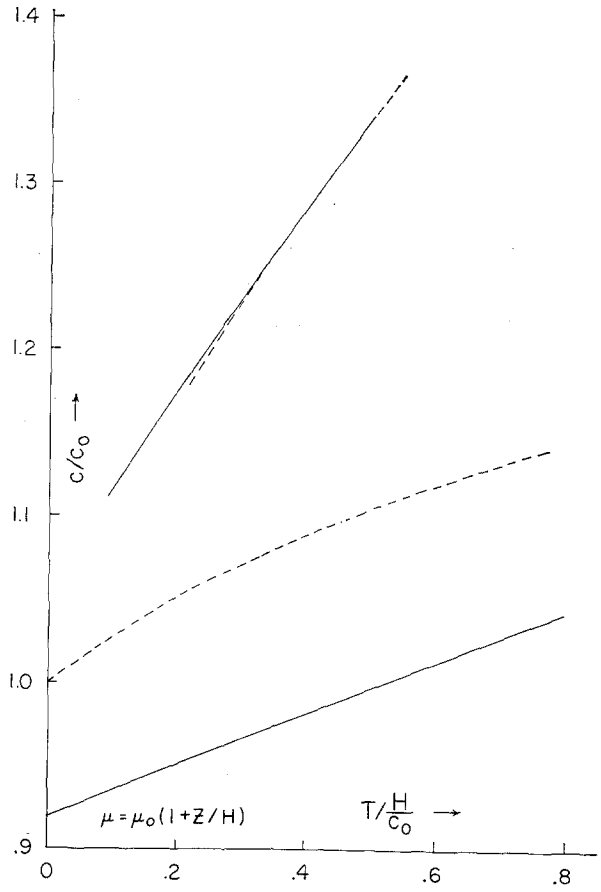

Fig. 8. Dispersion curves of Rayleigh waves propagated in a medium with constant density and linearly increasing rigidity. Broken lines are the dispersion curves of Love waves which were shown in figure 2.

If

$$\lambda = \mu, \qquad \text{or} \qquad \gamma^2 = 3 \tag{5.19}$$

we have

$$\begin{cases} \dfrac{d^2U}{d\zeta^2} + \{\overline{M}(\zeta) + \overline{K}(\zeta)\}\, U + 2\xi \dfrac{dW}{d\zeta} = 0 \\ \dfrac{d^2W}{d\zeta^2} + \{\overline{M}(\zeta) + \overline{L}(\zeta)\}\, W - \dfrac{2}{3}\xi \dfrac{dU}{d\zeta} = 0 \end{cases} \tag{5.20}$$

$$\overline{M}(\zeta) = \frac{1}{4\zeta^2}, \quad \overline{K}(\zeta) = \left(\frac{\mathfrak{v}^2}{\zeta} - 3\right)\xi^2 \quad \text{and} \quad \overline{L}(\zeta) = \left(\frac{\mathfrak{v}^2}{3\zeta} - \frac{1}{3}\right)\xi^2$$

and at $\zeta = 1$

$$\begin{cases} \xi W + \dfrac{dU}{d\zeta} - \dfrac{1}{2\zeta} U = 0 \\ -\xi U + 3\dfrac{dW}{d\zeta} - \dfrac{3}{2\zeta} W = 0 \end{cases} \tag{5.21}$$

Extra conditions (5.1) also hold at $\zeta = 1$.

5.2. Practical numerical computation

For the practical computation, formulae similar to (2.13) were employed. The fundamental equations are

$$\begin{cases} U_{n+1} = U_n + \Delta U_{n-1} + h^2 U_n'' + O(h^4) \\ \dfrac{d}{d\zeta} U_n = \dfrac{1}{6h} \{11\Delta U_{n-1} - 7\Delta U_{n-2} + 2\Delta U_{n-3} + O(h^4)\} \end{cases} \tag{5.22}$$

where h is the interval of numerical calculation,
Δ implies the difference.

We have similar formulae for the function W.

The final expressions for the calculation are

$$\begin{cases} U_{n+1} = U_n + \Delta U_{n-1} - h^2(\overline{M} + \overline{K})U_n - \dfrac{1}{3} h\xi(11\Delta W_{n-1} - 7\Delta W_{n-2} + 2\Delta W_{n-3}) \\ W_{n+1} = W_n + \Delta W_{n-1} - h^2(\overline{M} + \overline{L})W_n + \dfrac{1}{9} h\xi(11\Delta U_{n-1} - 7\Delta U_{n-2} + 2\Delta U_{n-3}) \end{cases} \tag{5.23}$$

$\overline{K}$, $\overline{L}$, and $\overline{M}$ are defined by (5.20). The derivatives at the free surface $\zeta = 1$ are given in the following expressions, which are necessary to start the numerical calculation:

$$\begin{cases} U = a\,, & W = 1 \\ \dfrac{dU}{d\zeta} = -\xi + \dfrac{1}{2} a\,, & \dfrac{dW}{d\zeta} = \dfrac{1}{3}\xi a + \dfrac{1}{2} \\ \dfrac{d^2U}{d\zeta^2} = -\left(\dfrac{1}{4} + \overline{K}\right) a - 2\xi \dfrac{dW}{d\zeta}\,, & \dfrac{d^2W}{d\zeta^2} = -\left(\dfrac{1}{4} + \overline{L}\right) + \dfrac{2}{3}\xi \dfrac{dU}{d\zeta} \\ \dfrac{d^3U}{d\zeta^3} = \left(\dfrac{1}{2} + \mathfrak{v}^2\xi^2\right) a - \left(\dfrac{1}{4} + \overline{K}\right) \dfrac{dU}{d\zeta} - 2\xi \dfrac{d^2W}{d\zeta^2} \\ \dfrac{d^3W}{d\zeta^3} = \left(\dfrac{1}{2} + \dfrac{1}{3}\mathfrak{v}^2\xi^2\right) - \left(\dfrac{1}{4} + \overline{L}\right) \dfrac{dW}{d\zeta} + \dfrac{2}{3}\xi \dfrac{d^2U}{d\zeta^2} \end{cases} \tag{5.24}$$

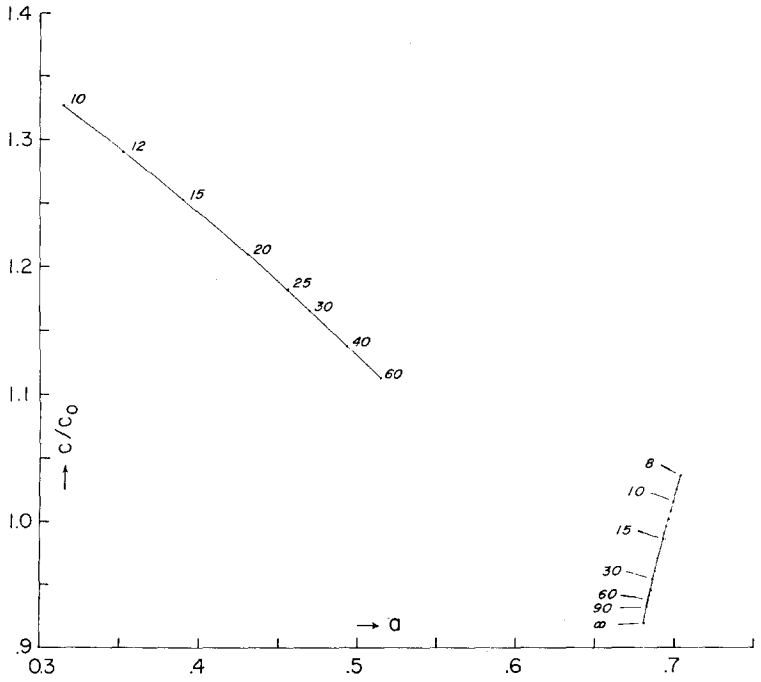

Fig. 9. Relation between υ and a which gives convergent distribution of U and W, for the fundamental mode, right, and the second mode, left. The parameter in the figure is ξ.

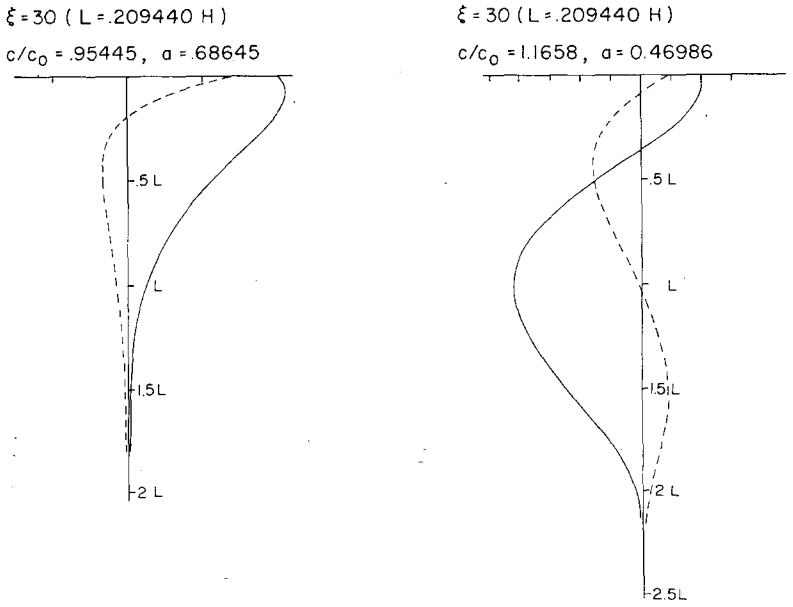

Fig. 10. Example of the distribution of U (broken line) and W (solid line) for the fundamental mode, left, and the second mode, right, for the heterogeneous half space.

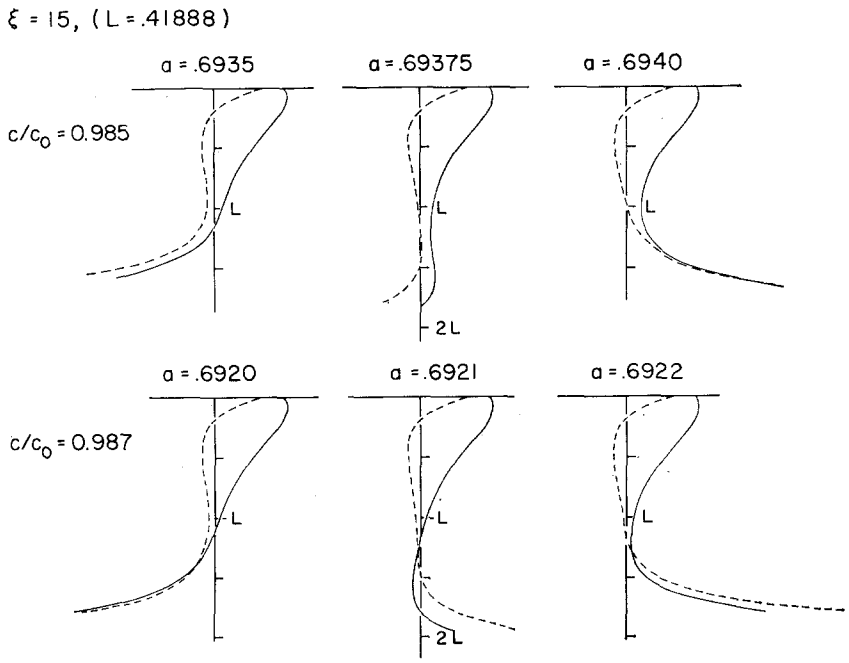

Fig. 11. Examples of the distribution of U and W which do not provide the convergent solution. For $\xi = 15$ the correct answer is $c/c_0 = 0.9861$, $a = 0.69280$.

The dispersion curves, first and second modes, are given in figure 8. Figure 9 gives the relation between phase velocity and the amplitude ratio a. Figure 10 is the example of the distribution of U and W.

The IBM "650" was also used for this calculation, and it took from thirty minutes to two hours to find one point with shorter times for better starting approximations.[8] In figure 11 typical examples of the distribution of the functions U and W are shown for various values of a and $\mathfrak{v}$ which do not give converging solutions.

In this paper a very simple structure of the medium was assumed for the computations. More complicated cases and the problem of the spherical elastic body must be the subject of future studies.

[8] In the present program ξ was fixed and $\mathfrak{v}$ and a were changed in order to get the answer. Although it is theoretically correct that at some definite combination of $\mathfrak{v}$ and a, U and W converge to zero, it is practically hard to find such a point sweeping the a–$\mathfrak{v}$ plane continually.

The practical method here adopted was to find the values of a and $\mathfrak{v}$ which make $U = W = 0$ at the depth $z = \epsilon L$, where L is a wave length and ϵ is a constant. Better approximations are obtained with larger values of ϵ. According to our experience and also from theoretical considerations, $\epsilon = 2$ gives a fairly good result, and $\epsilon = 3$ is large enough for the fundamental mode. For the second mode, however, sometimes we must make ϵ fairly large, say 5 or 6.

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